How to Measure the Spinor Character of Fermions

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A spinor is a geometrical object which changes its sign under a 2π-rotation. As fermions are described by spinors the question arises if spinors are mere mathematical tools or if they have physical reality. We propose an experiment from which the spinor character of fermions can directly be read. We take this opportunity to put the Stern-Gerlach experiment into the language of the quantum theoretical measuring process.

Introduction

The observable quantities of physical objects in classical physics are known to behave like scalars, vectors, or tensors of second or higher rank, which means that by subjecting the objects to an appropriate active transformation, their quantities transform like scalars, vectors, or tensors respectively.

As, however, in quantum theory fermions are described by wave functions which transform like spinors, the question arises if their spinoric character is observable, that is, if the sign change of the wave function under a 2π-rotation alters the results of a measurement. Aharonov and Susskind [1] have given a semi-classical gedankenexperiment, which suggests this observability.

We want to show here that within the framework of the quantum theoretical measuring process such an experiment can be constructed and therefore spinors are indeed observable. We will start with a rigorous treatment of the measurement process in Dirac formalism, which is based upon von Neumann's theory [2]. Then we shall apply it to the Stern-Gerlach experiment and show the peculiarities of this special process which partly completes and partly contrasts with Bohm's treatment [3]. Finally, these results will be used for the construction of our spinor experiment, which consists of two Stern-Gerlach magnets and a homogeneous magnetic field for one of the separated rays. In case of no homogeneous field (no rotation), the separation of the two rays is enlarged by the second magnet, whereas with rotation of one ray the interference term is enlarged and the direct terms tend to zero. We will therefore see different patterns on the photographic plate.

1. The Measuring Process

We call a combined system M + S a measurement if by looking at M we can make conclusions about the state of S. Especially, if the state of S is undisturbed by the process, it is called a measurement process "of the first kind". If it only determines the state of S before the interaction it is called "of the second kind". We shall divide the measurement process up into the four parts of the union, the interaction, the separation, and the section.

a) The Union of Measuring Apparatus and Object

If M is in the known state $|\Phi\rangle_M$ and S in the unknown state $|\varphi\rangle_S$, then one gets the state of the system M + S by taking the tensor-product of the two systems

$$|\Psi\rangle_{M+S} = |\Phi\rangle_M \otimes |\varphi\rangle_S .$$

Let A then be the quantity to be measured and $|\psi_i\rangle_S$ its representation base, then

$$|\varphi\rangle_S = \sum_i a_i |\psi_i\rangle_S ,$$

and as the tensor product is linear

$$|\Psi\rangle_{M+S} = \sum a_i |\Phi\rangle_M \otimes |\psi_i\rangle_S . \quad (1.1)$$

b) The Interaction

In the interaction picture, $|\Psi\rangle_{M+S}$ is a Heisenberg state and the interaction is described by the unitary operator

$$U_{int}(t) = \exp\{i H_{int} t\}$$

where H_{int} is the interaction Hamiltonian, and therefore

$$\begin{aligned} |\Psi(t)\rangle_{M+S} &= U_{int}(t) |\Psi\rangle_{M+S} \\ &= \sum_i a_i U_{int}(t) |\Phi\rangle_M \otimes |\psi_i\rangle_S . \end{aligned}$$

After the interaction, say $t > t'$, the Heisenberg state of the system will be

$$| \Psi' \rangle_{\mathrm{M+S}} \equiv | \Psi(t') \rangle_{\mathrm{M+S}} = \sum a_i \, U_{\mathrm{int}}(t') | \Phi \rangle_{\mathrm{M}} \otimes | \psi_i \rangle_{\mathrm{S}}$$

which, for a measurement of the first kind, has left the $| \psi_i \rangle_{\mathrm{S}}$ undisturbed

$$| \Psi' \rangle_{\mathrm{M+S}} = \sum a_i' \, | \Phi_i' \rangle_{\mathrm{M}} \otimes | \psi_i \rangle_{\mathrm{S}} . \quad (1.2)$$

c) The Separation

M is called an ideal measurement if now we can distinguish the states $| \Phi_i' \rangle_{\mathrm{M}}$ one from another, that is, if they are mutually orthogonal

$$_{\mathrm{M}}\langle \Phi_i' | \Phi_k' \rangle_{\mathrm{M}} = \delta_{ik} . \quad (1.3)$$

If this is not the case, as for the Stern-Gerlach experiment, the $| \Phi_i' \rangle_{\mathrm{M}}$ will interfere, and we will get no precise information about the state $| \psi_i \rangle_{\mathrm{S}}$. But still there can be found orthogonal states $| \xi_k \rangle_{\mathrm{M}}$ which correspond to orthogonal states $| \eta_k \rangle_{\mathrm{S}}$

$$| \Psi' \rangle_{\mathrm{M+S}} = \sum_k \beta_k | \xi_k \rangle_{\mathrm{M}} \otimes | \eta_k \rangle_{\mathrm{S}}$$

with real β_k and

$$_{\mathrm{M}}\langle \xi_i | \xi_k \rangle_{\mathrm{M}} = \delta_{ik} \quad \text{and} \quad _{\mathrm{S}}\langle \eta_i | \eta_k \rangle_{\mathrm{S}} = \delta_{ik}$$

so that by measuring $| \xi_k \rangle_{\mathrm{M}}$ one knows that S is in the mixed state $| \eta_k \rangle_{\mathrm{S}}$.

This bi-orthogonalisation is achieved by an antilinear mapping $F^{\#}$ from the Hilbert space of the object $\mathcal{H}_{\mathrm{S}}$ into the Hilbert space of the measuring apparatus $\mathcal{H}_{\mathrm{M}}$. It is defined by (compare [2-4]),

$$F^{\#} | \psi_i \rangle_{\mathrm{S}} = _{\mathrm{S}}\langle \psi_i | \Psi' \rangle_{\mathrm{M+S}} ,$$
$$F | \Phi_i' \rangle_{\mathrm{M}} = _{\mathrm{M}}\langle \Phi_i' | \Psi' \rangle_{\mathrm{M+S}} .$$

Note that

$$_{\mathrm{S}}\langle \psi_i | \Psi' \rangle_{\mathrm{M+S}} \in \mathcal{H}_{\mathrm{M}} \quad \text{and} \quad _{\mathrm{M}}\langle \Phi_i' | \Psi' \rangle_{\mathrm{M+S}} \in \mathcal{H}_{\mathrm{S}} .$$

For the orthogonal states we then have

$$| \eta_k \rangle_{\mathrm{S}} = \frac{1}{\beta_k} F | \xi_k \rangle_{\mathrm{M}} ,$$
$$F F^{\#} | \xi_k \rangle_{\mathrm{M}} = \beta_k^2 | \xi_k \rangle_{\mathrm{M}} ,$$
$$F^{\#} F | \eta_k \rangle_{\mathrm{S}} = \beta_k^2 | \eta_k \rangle_{\mathrm{S}} .$$

The separation of the two systems is established by a reduction to the measuring system M, which has the statistical operator

$$W_{\mathrm{M}} = F F^{\#}$$

and the probability w_k to find $| \xi_k \rangle_{\mathrm{M}}$ — and therefore to measure $| \eta_k \rangle_{\mathrm{S}}$ — is

$$w_k = \beta_k^2 .$$

d) The Section

Forgetting about how we get to know about the state of the measuring apparatus has been called the "section" [5]. As it can arbitrarily be shifted along the row of measuring processes, we shall cut as close to the object to be measured on as possible. We will therefore not be concerned with the problem of how we get to know the state $| \xi_k \rangle_{\mathrm{M}}$. This is what we mean by saying "we look at $| \xi_k \rangle_{\mathrm{M}}$".

2. The Stern-Gerlach Experiment

The Stern-Gerlach experiment is a device for a spin measurement of, say, a spin-$\frac{1}{2}$ particle. The particle passes through an inhomogeneous magnetic field where a momentum in positive or negative
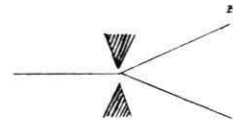
z-direction, depending on its spin direction, is transferred to the particle. This means that we measure the spin by looking at the location of the particle after the interaction; the location of the particle therefore serves as measuring apparatus M:

$\mathcal{H}_{\mathrm{M}} = $ location space, $\mathcal{H}_{\mathrm{S}} = $ spin space, and the wave function (1.1) becomes

$$| \Psi \rangle = | f(z) \rangle \otimes (a_1 | + \rangle + a_2 | - \rangle)$$

where $| + \rangle$, $| - \rangle$ are the spin eigenfunctions for the z-component of the spin and $| f(z) \rangle$ is a localized and normalized wave packet

$$| f(z) \rangle = \int f(k) | k \rangle \, \mathrm{d}k .$$

The interaction Hamiltonian is

$$\mathcal{H}_{\mathrm{int}} = - \boldsymbol{\sigma} \, \boldsymbol{B} .$$

The magnetic field $\boldsymbol{B}$ shall only have a nonvanishing z-component

$$B_z = B_0 + B_1 \, z .$$

The interaction operator then is

$$U_{\mathrm{int}}(t) = \exp\{ - i (B_0 + B_1 \, z) \sigma_z \, t \} .$$

The state (1.2) after the interaction is

$$| \Psi' \rangle = a_1 | f_+ \rangle \otimes | + \rangle + a_2 | f_- \rangle \otimes | - \rangle \quad (2.1)$$

where

$$| f_+ \rangle = \exp\{ - i (B_0 + B_1 \, z) t' \} | f(z) \rangle .$$
$$| f_- \rangle = \exp\{ i (B_0 + B_1 \, z) t' \} .$$

As the orthogonality condition (1.3) is not fulfilled

$$\langle f_+ | f_- \rangle = \exp\{2\,i(B_0 + B_1\,z)\,t'\} \qquad (2.2)$$

we construct the orthogonal states

$$|\,\eta_1\rangle = a_1\,|+\rangle + a_2\,|-\rangle \,,$$
$$|\,\eta_2\rangle = a_2{}^*\,|-\rangle - a_1{}^*\,|-\rangle \,,$$
$$|\,\xi_1\rangle = \frac{1}{\sqrt{w_1}}\,(a_1{}^*\,a_1\,|\,f_+\rangle + a_2{}^*\,a_2\,|\,f_-\rangle) \,,$$
$$|\,\xi_2\rangle = \frac{1}{\sqrt{w_2}}\,(a_2\,a_1\,|\,f_+\rangle - a_1\,a_2\,|\,f_-\rangle)$$

where

$$a_1 = \frac{a_1\,a_2{}^* \langle f_- | f_+ \rangle}{w_1 - |\,a_1\,|^2}\,a_2 \,, \qquad a_2 = \frac{a_1{}^*\,a_2 \langle f_+ | f_- \rangle}{w_1 - |\,a_1\,|^2}$$

or

$$w_1{}^2 - w_1 + |\,a_1\,a_2\,|^2(1 - |\,\langle f_+ | f_- \rangle\,|^2) = 0 \,.$$

After adding the normalization conditions

$$|\,a_1\,|^2 + |\,a_2\,|^2 = 1 \,, \quad |\,a_1\,|^2 + |\,a_2\,|^2 = 1 \,, \quad w_1 + w_2 = 1 \,,$$

the rest is a matter of computation.

Because of (2.2)

$$|\,\langle f_+ | f_- \rangle\,|^2 = 1$$
$$w_1 = 1, \quad w_2 = 0, \quad |\,a_1\,| = |\,a_1\,|, \quad |\,a_2\,| = |\,a_2\,| \,.$$

This shows that the interference is maximal and the two-spin directions cannot be distinguished yet. Taking now the time evolution of the wave packet — which means that we change to the Schrödinger-picture — the two wave packets $|\,f_+(t)\rangle$ and $|\,f_-(t)\rangle$ will separate as they have opposite momentum in z-direction. (For simplicity we set $t = 0$ after the interaction.)

$$|\,f_\pm(t)\rangle = \int f(k)\exp\{i\,E_{\mathrm{kin}}^\pm t\}$$
$$\cdot \exp\{\mp i(B_0 + B_1\,z)\,t'\}\,|\,k\rangle\,\mathrm{d}k \,,$$
$$E_{\mathrm{kin}}^\pm(k) = \frac{1}{2\,m}\,(k \mp B_1\,t')^2$$

which yields the group velocities

$$v_{\mathrm{G}}^\pm = \frac{\mathrm{d}E_{\mathrm{kin}}^\pm(k)}{\mathrm{d}k}\bigg|_{k=0} = \mp\frac{B_1\,t'}{m}$$

for positive and negative spin particles, respectively. Therefore

$$\langle f_+(t)|\,f_-(t)\rangle = \exp\{2\,i\,(B_0 + B_1\,z)\,t'\} \qquad (2.3)$$
$$\cdot \int |\,f(k)\,|^2 \exp\{-i\,(2\,B_1\,t'/m)\,k\,t\}\,\mathrm{d}k \,.$$

For a homogeneous magnetic field $B_1 = 0$, and the interference term remains constant. For the inhomo-

geneous field (2.3) tends to zero as is easiest seen by setting for the wave packet

$$f(k) + A\,e^{-(k/a)^2}$$

which yields

$$\langle f_+(t)|\,f_-(t)\rangle = \exp\{2\,i(B_0 + B_1\,z)\,t'\}$$
$$\cdot \exp\left\{-\frac{B_1\,t'}{m}\,a^2\,t\right\}\,k\,t$$

which goes exponentially towards zero with time. The separation of the centers of the two wave packets is given by

$$\Delta s(t) = |\,v_{\mathrm{G}}^+\,t - v_{\mathrm{G}}^-\,t\,| = 2_1\frac{|\,B_1\,|\,t'}{m}\,t \,.$$

These considerations show that there are no quantitative restrictions to the momentum transfer (in contrast to [3]), although, of course, for large momentum transfer ($|\,B_1\,|\,t'/m$ large) and a well localized wave packet (a large), the interference term vanishes faster.

When the interference is practically zero, then

$$w_1 = |\,a_1\,|^2, \quad w_2 = |\,a_2\,|^2, \quad |\,a_1\,| = 1, \quad |\,a_2\,| = 0$$

and then the Stern-Gerlach experiment is suited for a spin measurement and the measurement is an ideal measurement of the first kind.

3. Measurement of the Spinor Character

A homogeneous magnetic field does not alter the momentum of a spin-particle, but is rotates the particle according to its spin direction.

In order to measure the spinor character of the particles, we will — after the particles have passed through a Stern-Gerlach magnet — rotate one of the two orthogonal states, say $|\,\xi_1\rangle \otimes |\,\eta_1\rangle$, which shall be established by a homogeneous magnetic field. Then

$$U_{\mathrm{int}}(t) = \exp\{-i\,\sigma_z\,B_0\,t\}\,P_{[\,|\,\xi_1\rangle \otimes |\,\eta_1\rangle\,]}$$
$$= R(2\,\sigma_z\,B_0\,t)\,P_{[\,|\,\eta_1\rangle\,]}$$

where $R(\vartheta)$ is the rotation operator. If we now let the interaction last till $t = t'$, so that

$$B_0\,t' = \pi$$

we have a $2\,\pi$-rotation of $|\,\eta_1\rangle$ and therefore a sign change

$$R(2\,\pi)\,|\,\eta_1\rangle = -|\,\eta_1\rangle \,,$$
$$|\,\Psi''\rangle \equiv R(2\,\pi)\,P_{[\,|\,\eta_1\rangle\,]}\,|\,\Psi'\rangle = -w_1\,|\,\xi_1\rangle \otimes |\,\eta_1\rangle$$
$$= w_2\,|\,\xi_2\rangle \otimes |\,\eta_2\rangle \,.$$

Setting $|\bar{\eta}_1\rangle = -|\eta_1\rangle$ we get

$$|\bar{\xi}_1\rangle = \frac{1}{\sqrt{w_1}}\, F^{\#}\,|\bar{\eta}\rangle = -|\xi_1\rangle$$

and therefore

$$|\Psi''\rangle = w_1\,|\bar{\xi}_1\rangle \otimes |\bar{\eta}_1\rangle + w_2\,|\xi_2\rangle \otimes |\eta_2\rangle$$

which shows that the statistical operators of the sub-systems have not changed. But still this state differs from (2.1)

$$|\Psi''(t)\rangle = (|a_1|^2 - |a_2|^2)\,(-a_1\,|f_+(t)\rangle \otimes$$
$$|+\rangle + a_2\,|f_-(t)\rangle \otimes|-\rangle) + |\Psi''_{\mathrm{int}}(t)\rangle,$$

$$|\Psi''_{\mathrm{int}}(t)\rangle = -2\,(a_1\,a_2^*\,a_2\,|f_-(t)\rangle \otimes |+\rangle$$
$$+ a_1^*\,a_2\,a_1\,|f_+(t)\rangle \otimes |-\rangle).$$

For computational simplicity we will, from now on, choose $|\alpha_1|^2 = |\alpha_2|^2 = \frac{1}{2}$ and therefore also $|a_1|^2 = |a_2| = \frac{1}{2}$, then the direct terms vanish and we are left with the interference term

$$|\bar{\Psi}''(t)\rangle \equiv -|\Psi''_{\mathrm{int}}(t)\rangle .$$

Again we construct the bi-orthogonal representation

$$|\eta_1''\rangle = b_1\,|+\rangle + b_2\,|-\rangle,$$
$$|\xi_1''\rangle = 2\,b_1^*\,a_1\,a_2^*\,a_2\,|f_-(t)\rangle$$
$$+ 2\,b_2^*\,a_1^*\,a_2\,a_1\,|f_+(t)\rangle$$

and analogously for $|\eta_2''\rangle$ and $|\xi_2''\rangle$, which yields

$$b_1 = a_2 \quad \text{and} \quad b_2 = a_1$$

and the probabilities

$$w_1'' = w_1 \quad \text{and} \quad w_2'' = w_2 .$$

As stated before, the statistical operator has not changed, but instead of finding a particle with posi-tive spin, we now find one with negative spin, which we do not realize as we are only looking at the loca-tion of the particle. A subsequent Stern-Gerlach experiment should therefore show the difference.

Following through our notation the state after the interaction is

$$|\bar{\Psi}'''\rangle = 2\,(a_1\,a_2^*\,a_2\,|f_-(t)_+\rangle \otimes |+\rangle$$
$$+ a_1^*\,a_2\,a_1\,|f_+(t)_-\rangle \otimes |-\rangle)$$

whereas without rotation it would have been

$$|\Psi'''\rangle = a_1\,|f_+(t)_+\rangle \otimes |+\rangle + a_2\,|f_-(t)_-\rangle \otimes |-\rangle .$$

In the first case the separation of the two rays is diminished; in the second case it is increased, which is schematically shown in the following Figure:

without rotation   with rotation

This shows that, if a homogeneous field corres-ponding to a 2π-rotation is put between two Stern-Gerlach magnets, the pattern on the screen behind the second magnet changes when the field is turned on or off. We can therefore conclude that spinors have physical reality as well as scalars, vectors or tensors of second or higher rank.

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